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Hamiltonian formulation for solitary waves propagating on a variable background

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Abstract. Solitary waves propagating on a variable background are conventionally described by the variable-coefficient Korteweg-deVries equation. However, the underlying physical system is often Hamiltonian, with a conserved energy functional. Recent studies for water waves and interfacial waves have shown that an alternative approach to deriving an appropriate evolution equation, which asymptotically approximates the Hamiltonian, leads to an alternative variable-coefficient Korteweg-deVries equation, which conserves the underlying Hamiltonian structure more explicitly. This paper examines the relationship between these two evolution equations, which are asymptotically equivalent, by first discussing the conservation laws for each equation, and then constructing asymptotically a slowly-varying solitary wave.

Key words: solitary waves, Korteweg-deVries, Hamiltonian systems.

1. Introduction

It is now well-known that one of the principal canonical equations to describe weakly nonlinear long waves is the Korteweg-deVries (KdV) equation, which has the familiar solitary wave as one of its primary solutions. When the background in which this solitary wave is propagating is not uniform, but instead is variable on some long length-scale, then it is also well-known that the constant-coefficient KdV equation is replaced by the variable-coefficient KdV equation

$$u_\tau + \frac{c_\tau}{2c}u + \frac{\mu}{c}uu_\xi + \frac{\lambda}{c^3}u_{\xi\xi\xi} = 0. \tag{1.1}$$

Here the independent variables are

$$\tau = \varepsilon^2 x, \qquad \xi = \frac{1}{\varepsilon^2}\int_0^\tau \frac{\mathrm{d}\tau'}{c(\tau')} - t, \tag{1.2}$$

and the coefficient c, μ and λ are all functions of τ. Here c is the linear long-wave phase speed of the underlyng physical system, while μ and λ are the nonlinear and dispersive coefficients, respectively, and are also determined by the linear long-wave structure of the underlying physical system. The independent variables x and t represent space and time, respectively, in a suitable long-wave non-dimensionalization, and ε is a small parameter such that ε^2 measures the effects of both nonlinearity and dispersion. Equations of the form (1.1) have been derived for solitary water waves by Ostrovsky and Pelinovsky [1], Kakutani [2] and Johnson [3], and by Grimshaw [4] (see also [5] for a summary) for internal solitary waves.

Recently, however, van Groesen and Pudjaprasetya [6] (see also [7] have pointed out that, for the case of solitary water waves, the underlying physical system is Hamiltonian, with a conserved Hamiltonian functional representing energy. They exploited this to derive an alternative to the conventional variable-coefficient KdV equation (1.1), in that the energy is approximated more accurately than in (1.1), and which conserves the underlying Hamiltonian structure more explicitly. A similar result was obtained for interfacial waves in [8]. The new Hamiltonian variable-coefficient KdV equation is formulated in terms of the Hamiltonian

$$H = \int_{-\infty}^{\infty} J \,\mathrm{d}x, \tag{1.3a}$$

where

$$J = \tfrac{1}{2}u^2 + \varepsilon^2\{-(\tfrac{1}{2}\lambda)u_x^2 + (\tfrac{1}{6}\mu)u^3\}. \tag{1.3b}$$

Here H is an approximation to the full Hamiltonian of the underlying physical system, with an error of $0(\varepsilon^4)$. The evolution equation is then

$$u_t = -\Gamma\frac{\delta H}{\delta u}, \tag{1.4a}$$

where

$$\Gamma = \frac{1}{2}\left\{c\frac{\partial}{\partial x} + \frac{\partial}{\partial x}c\right\}, \tag{1.4b}$$

or

$$u_t + \left\{c\frac{\partial}{\partial x} + \tfrac{1}{2}c_x\right\}\{u + \varepsilon^2\lambda u_{xx} + \varepsilon^2(\tfrac{1}{2}\mu)u^2\} = 0. \tag{1.4c}$$

The skew-symmetric operator in (1.4a) ensures that this equation is Hamiltonian, and conserves the Hamiltonian H. Although evolution equations of the form (1.4a) have so far only been obtained for the special case of solitary water and interfacial waves, we conjecture that (1.4) is an alternative Hamiltonian formulation of (1.1) in all cases when (1.1) holds. In support of this we note that (1.4a) and (1.1) are asymptotically equivalent under the transformation (1.2), which converts (1.4a) into the equation

$$u_\tau + \frac{c_\tau}{2c}u + \left\{\frac{1}{c}\frac{\partial}{\partial\xi} + \varepsilon^2\left(\frac{\partial}{\partial\tau} + \frac{c_\tau}{2c}\right)\right\}M = 0,$$

where

$$M = \frac{\lambda}{c^2}u_{\xi\xi} + (\tfrac{1}{2}u)u^2 + \varepsilon^2\left(\frac{\lambda}{c}\right)_\tau u_\xi + 2\varepsilon^2\frac{\lambda}{c}u_{\xi\tau} + O(\varepsilon^4). \tag{1.5}$$

This reduces to (1.1) with the omission of $O(\varepsilon^2)$ terms.

In this paper, we explore further the relationship between the two alternative variable-coefficient KdV equations (1.1) and (1.4). In Section 2, we examine the conservation laws of the two equations, and the relationships between them. In Section 3 we construct the asymptotic slowly-varying solitary wave solution of (1.4), and compare it with the corresponding solution of (1.1). We conclude with some discussion in Section 4.

2. Conservation laws

The Hamiltonian variable-coefficient KdV equation (1.4a) conserves the Hamiltonian itself, by virtue of the skew-symmetry of the operator Γ, and so

$$E = \int_{-\infty}^{\infty} J \, \mathrm{d}x = \text{constant}, \tag{2.1}$$

where the density J is defined by (1.3b). Here, of course, E is just the Hamiltonian H, but we have renamed it here, since in many physical applications it corresponds to the energy. Equation (1.4a) also conserves the 'mass'

$$M = \int_{-\infty}^{\infty} \frac{u}{\sqrt{c}} \, \mathrm{d}x = \text{constant}. \tag{2.2}$$

We note that, although M is an exact invariant for (1.4a), it usually differs from the exact expression for mass in the underlying physical system by terms of $O(\varepsilon^2)$, due to the generation of left-going waves (see, for instance, [8–10]). Because of the explicit dependence of c on τ, Equation (1.4a) will not, in general, possess any other invariants. In particular, the 'momentum'

$$P = \int_{-\infty}^{\infty} \frac{1}{2} \frac{u^2}{c} \, \mathrm{d}x, \tag{2.3}$$

is not conserved. Instead, we find that

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\varepsilon^2 \int_{-\infty}^{\infty} \frac{c_\tau}{2c} u^2 \, \mathrm{d}x + O(\varepsilon^4) \tag{2.4}$$

Next we turn to the standard variable-coefficient KdV equation (1.1). It is readily shown that this has two invariants

$$M^* = \int_{-\infty}^{\infty} \sqrt{c} u \, \mathrm{d}\xi = \text{constant}, \tag{2.5a}$$

and

$$E^* = \int_{-\infty}^{\infty} \tfrac{1}{2} c u^2 \, \mathrm{d}\xi = \text{constant}. \tag{2.5b}$$

Note that M^* and E^* are the counterparts of the expressions M (2.2) and E (2.1), respectively, since, to leading order in ε^2, the transformation (1.2) implies that $\mathrm{d}x = c \, \mathrm{d}\xi$. Again, it would seem that there are, in general, no other invariants of (1.1), and in particular, the momentum

$$P^* = \int_{-\infty}^{\infty} \tfrac{1}{2} u^2 \, \mathrm{d}\xi \tag{2.6}$$

is not conserved. Instead, we find that

$$\frac{\mathrm{d}P^*}{\mathrm{d}\tau} = -\int_{-\infty}^{\infty} \frac{1}{2}\frac{c_\tau}{c} u^2 \,\mathrm{d}\xi. \tag{2.7}$$

Of course, (2.6) and (2.7) are just the counterparts of (2.2) and (2.4), respectively.

It is interesting to observe that (1.1) is also a Hamiltonian system in the sense that it can be expressed in the form

$$u_\tau + \frac{c_\tau}{2c} u = -\frac{1}{c}\frac{\partial}{\partial\xi}\frac{\delta H^*}{\delta u}, \tag{2.8a}$$

where

$$H^* = \int_{-\infty}^{\infty} K^* \,\mathrm{d}\xi, \tag{2.8b}$$

and

$$K^* = -\frac{\lambda}{2c^2} u_\xi^2 + (\tfrac{1}{6}\mu) u^3. \tag{2.8c}$$

However, because K^* depends explicitly on τ, H^* is not a conserved quantity. It is pertinent to note here that even though $\frac{1}{2}u^2 + \varepsilon^2 K^*$ is just the Hamiltonian density J (with an error of $O(\varepsilon^2)$), the corresponding quantity $E^* + \varepsilon^2 c H^*$ is not conserved by Equation (1.1), or its extended form (1.5). The reason for this is that the $O(\varepsilon^2)$ terms in the transformation from the variable x in the integrand of (2.1) to the variable ξ in the integrand of (2.5b) must be taken into account.

Indeed, it is clear that the energy E (2.1), being a conserved quantity for (1.4), should remain a conserved quantity to $O(\varepsilon^4)$ at least, under the transformation (1.2). Thus, we see that

$$J(x, t) = \widehat{J}(\varepsilon^2 s, \xi), \tag{2.9a}$$

where

$$s(\tau) = \frac{1}{\varepsilon^2}\int_0^\tau \frac{\mathrm{d}x'}{c(\tau')}. \tag{2.9b}$$

Hence, noting that $\xi = s - t$ (1.2), we find

$$E = \int_{-\infty}^{\infty} J(x, t)\,\mathrm{d}x = \int_{-\infty}^{\infty} \widehat{J}(\varepsilon^2(\xi + t), \xi) c \,\mathrm{d}\xi. \tag{2.10}$$

Here we are, for convenience, temporarily using $\widehat{\tau} = \varepsilon^2 s$ in place of τ. Next, we expand the integrand in (2.10) as follows, so that

$$E = \int_{-\infty}^{\infty} \widehat{J}(\varepsilon^2 t, \xi) c \,\mathrm{d}\xi + \varepsilon^2 \int_{-\infty}^{\infty} \xi (c\widehat{J})_{\widehat{\tau}} \,\mathrm{d}\xi + \cdots. \tag{2.11}$$

Then, on reverting to τ in place of $\hat{\tau}$ in (2.11), we can conclude that the correct expression for the energy E with respect to Equation (1.1) is

$$E = \int_{-\infty}^{\infty} J(\tau,\xi)c \, d\xi + \varepsilon^2 \int_{-\infty}^{\infty} (cJ)_\tau c\xi \, d\xi + O(\varepsilon^4). \tag{2.12}$$

From (1.3b), and again using the transformation (1.2), we see that

$$J(\tau,\xi) = \tfrac{1}{2}u^2 + \varepsilon^2\left(-\frac{\lambda}{2c^2}u_\xi^2 + (\tfrac{1}{6}\mu)u^3\right) + O(\varepsilon^4). \tag{2.13}$$

Next, we utilize Equation (1.1) to evaluate $(cJ)_\tau$ in (2.12) and finally obtain the result that

$$E = \int_{-\infty}^{\infty} \tfrac{1}{2}cu^2 \, d\xi + \varepsilon^2 \int_{-\infty}^{\infty} \left(-\frac{2\lambda}{c^2}u_\xi^2 + (\tfrac{1}{2}\mu)u^3\right) c \, d\xi + O(\varepsilon^4). \tag{2.14}$$

The first term here is just E^* (2.5b), and this is conserved by Equation (1.1). It can now be verified that the full expression (2.14) is conserved to $O(\varepsilon^4)$ by the extended form of Equation (1.1), namely Equation (1.5). It is now also clear that, while (1.4) conserves the correct expression E (2.1) for the energy to an error of $O(\varepsilon^4)$ (recall that H (1.3) is an approximation to the full Hamiltonian with an $O(\varepsilon^4)$ error), the traditional Equation (1.1) only conserves E to an error of $O(\varepsilon^2)$.

3. Slowly-varying solitary wave

In this section we construct the slowly-varying solitary-wave solution of the Hamiltonian variable-coefficient KdV equation (1.4a), in order to compare the result with the standard well-known corresponding theory for the variable-coefficient KdV equation (1.1). Thus here we let

$$s = \sigma x \tag{3.1}$$

and assume that c, λ and μ are all functions of s. Here we must assume that $\sigma \ll \varepsilon^2$, in order that the terms arising from the slow variation are smaller than the $O(\varepsilon^2)$ terms in (1.4a). Indeed, we now adopt the point of view that, even though ε^2 is a small parameter, we shall regard (1.4a) as a given 'exact' equation. Strictly, we should also assume that $\sigma \ll \varepsilon^4$ so that terms of $O(\sigma)$ are greater than the $O(\varepsilon^4)$ error terms in (1.4a). Note also that in terms of τ (1.2)

$$s = \beta\tau, \qquad \beta = \sigma/\varepsilon^2 \tag{3.2}$$

and we are assuming that $\beta \ll 1$.

The asymptotic procedure for constructing the slowly-varying solitary wave is standard as far as (1.1) is concerned (see, for instance, Grimshaw and Mitsudera, [11]), so here we give just a brief outline. We seek an asymptotic solution of (1.4a) whose leading term is a solitary wave of variable amplitude $a(s)$ and variable speed $V(s)$. Thus we put

$$\theta = \frac{1}{\sigma}\int_0^{'s} \frac{ds'}{V(s')} - t, \tag{3.3}$$

and seek a solution of (1.4a) of the form

$$u = u_0(\theta, s) + \sigma u_1(\theta, s) + O(\sigma^2), \tag{3.4a}$$

$$V = V_0(s) + \sigma V_1(s) + O(\sigma^2). \tag{3.4b}$$

It is readily found that u_0 satisfies the equation

$$\frac{(V_0 - c)}{c} u_0 = \varepsilon^2 \left(\frac{\lambda}{V_0^2} u_{0\theta\theta} + (\tfrac{1}{2}\mu) u_0^2 \right), \tag{3.5}$$

which has the well-known solitary-wave solution

$$u_0 = a \operatorname{sech}^2 \gamma\theta, \tag{3.6a}$$

where

$$\frac{V_0 - c}{c} = \varepsilon^2 \frac{\mu a}{3} = 4\varepsilon^2 \lambda \left(\frac{\gamma}{V_0} \right)^2. \tag{3.6b}$$

At the next order we obtain the following equation for u_1

$$\frac{c}{V_0} \frac{\partial}{\partial\theta} \left\{ -\frac{(V_0 - c)}{c} u_1 + \varepsilon^2 \left(\frac{\lambda}{V_0^2} u_{1\theta\theta} + \mu u_0 u_1 \right) \right\} + F_1 = 0, \tag{3.7a}$$

where

$$\begin{aligned} F_1 \;=\; & (V_0 u_0)_s - \frac{c_s}{2c}(V_0 u_0) + \frac{\varepsilon^2 c}{V_0} \frac{\partial}{\partial\theta} \left\{ \frac{2\lambda}{V_0} U_{0\theta s} + \left(\frac{\lambda}{V_0} \right)_s u_{0\theta} \right\} \\ & + V_1 \left\{ -\frac{c}{V_0^2} u_{0\theta} + \varepsilon^2 \left[-\frac{3c\lambda}{V_0^4} u_{0\theta\theta\theta} - \frac{c\mu}{V_0^2} u_0 u_{0\theta} \right] \right\}. \end{aligned} \tag{3.7b}$$

Here we have used (3.5) to simplify the expression for F_1. The compatibility condition for (3.7a) is

$$\int_{-\infty}^{\infty} F_1 u_0 \, \mathrm{d}\theta = 0, \tag{3.8}$$

and this then yields the equation

$$\frac{\partial}{\partial s} \left\{ \int_{-\infty}^{\infty} \frac{1}{2} \frac{V_0^2 u_0^2}{c} \, \mathrm{d}\theta - \varepsilon^2 \int_{-\infty}^{\infty} \frac{\lambda}{V_0} u_{0\theta}^2 \, \mathrm{d}\theta \right\} = 0, \tag{3.9a}$$

and so

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{V_0^2 u_0^2}{c} \, \mathrm{d}\theta - \varepsilon^2 \int_{-\infty}^{\infty} \frac{\lambda}{V_0} u_{0\theta}^2 \, \mathrm{d}\theta = \text{constant}. \tag{3.9b}$$

This last expression thus determines the variation of the amplitude $a(s)$. Before substituting (3.6a) in (3.9b) to achieve this, we note that (3.9b) can also be written in the form (using (3.5)),

$$\int_{-\infty}^{\infty} \tfrac{1}{2} V_0 u_0^2 \, \mathrm{d}\theta + \varepsilon^2 \int_{-\infty}^{\infty} \left(-\frac{\lambda}{2V_0^2} u_{0\theta}^2 + \frac{\mu}{6} u_0^3 \right) V_0 \, \mathrm{d}\theta = \text{constant}. \tag{3.10}$$

Noting that $\mathrm{d}x = V\,\mathrm{d}\theta$ for a fixed value of t (see (3.3), we see that the left-hand side of (3.10) is just the energy E (2.1) evaluated for the leading-order solitary-wave term u_0 (3.6a). Thus, as expected, the variation of the solitary wave amplitude is determined by the conservation of energy.

The analogous theory for the conventional KdV equation (1.1) leads to the same expression (3.10), but with the omission of the $O(\varepsilon^2)$ term on the left-hand side of (3.10), the replacement of V_0 with c, and $\mathrm{d}\theta$ with $\mathrm{d}\xi$, again with errors of $O(\varepsilon^2)$. Indeed, this is to be expected, since the conservation of energy for (1.1) is given by (2.5b). In making this comparison, we note that the transformation of variables (1.2) implies that for (1.1) u_0 is again given by (3.6a) which can now be written as

$$u_0 = a\,\mathrm{sech}^2\gamma\theta, \tag{3.11a}$$

where

$$\theta = \xi - \frac{1}{\beta}\int_0^s W(s')\,\mathrm{d}s', \tag{3.11b}$$

and

$$W = \frac{1}{\varepsilon^2}\frac{(V-c)}{Vc}. \tag{3.11c}$$

Using (3.6b) we see that, to leading order in ε^2, this last relation becomes

$$W_0 = \frac{\mu a}{3c} = \frac{4\lambda}{c^3}\gamma^2. \tag{3.12}$$

Thus u_0, given by (3.11a), reduces to the well-known solitary-wave solution of the conventional KdV equation (1.1).

Next we substitute the expressions (3.6a) for u_0 in (3.10) and obtain,

$$|a|^{3/2}\left(\frac{\lambda}{|\mu|}\right)^{1/2}\left\{1+\varepsilon^2\frac{\mu a}{5}\right\} = \text{constant}. \tag{3.13}$$

This is the required expression for the variation of the amplitude $a(s)$. Remarkably, it depends only on λ and μ, and not on c, although, of course, V_0 and γ do depend on c as well. To leading order in ε^2, (3.13) reduces to the well-known result for the standard variable-coefficient-KdV equation (1.1). Further, we see from (3.13) that the $O(\varepsilon^2)$ term is significant in practice only when μ^4/λ becomes relatively large. Indeed, to emphasise this (3.13) can be written in the form

$$A\left\{1+\frac{\varepsilon^2 A}{5}\frac{|\mu|^{4/3}}{\lambda^{1/3}}\right\} = \text{constant}, \tag{3.14a}$$

where

$$A = |a|^{3/2}\left(\frac{\lambda}{|\mu|}\right)^{1/2}. \tag{3.14b}$$

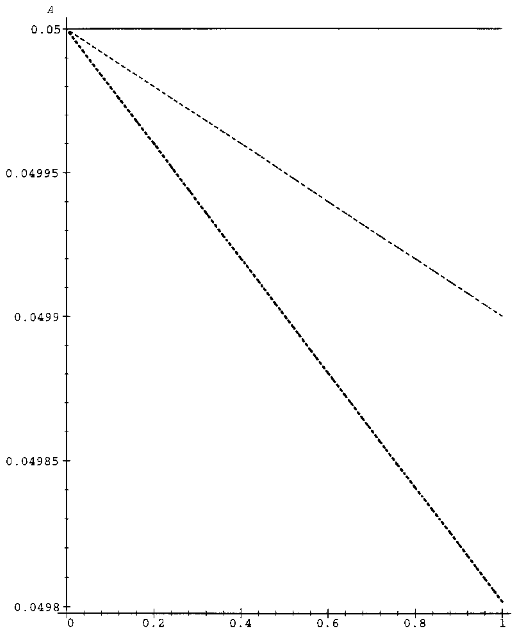

Figure 1. A plot of $A = |a|^{3/2}(\lambda/|\mu|)^{1/2}$ (see (3.24)) as a function of $|\mu|^{4/3}/\lambda^{1/3}$ for various values of ε; $\varepsilon = 0$ (top line), 0·3 (middle line, ---) and 0·6 (bottom line).

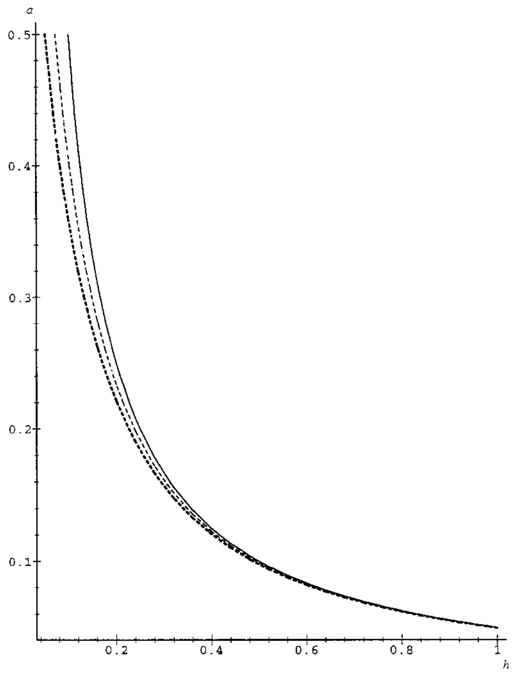

Figure 2. A plot of a as a function of h (see (3.15)) for various values of ε; $\varepsilon = 0$ (top line), 0·3 (middle line, ---) and (0·6) (bottom line).

The expression (3.14a) is plotted as a function of $|\mu^{4/3}/\lambda^{1/3}$ for various values of ε^2 in Figure 1. It is interesting to note that the effect of the $O(\varepsilon^2)$ term is always to reduce the value of A.

As an illustration, let us consider the case of a solitary wave propagating over variable depth h. In this case $\lambda = \frac{1}{6}h^2$ and $\mu = 3/2h$ while a is the amplitude of the solitary wave above the undisturbed level, and so (3.13) reduces to

$$(ah)^{3/2}\left\{1 + \varepsilon^2 \frac{3a}{10h}\right\} = \text{constant}. \tag{3.15}$$

To leading order in ε^2, this gives the well-known result that $a \propto h^{-1}$. The effect of the $O(\varepsilon^2)$ term is shown in Figure 2, where we plot a as a function of h for various values of ε^2. We see that the effect of this term is to reduce the amplitude growth, particularly as $h \to 0$.

Next we recall the well-known property that, although the slowly-varying solitary wave conserves energy, it cannot by itself conserve mass (M, see (2.2)). Instead, this is conserved by the creation of a trailing shelf of amplitude $O(\sigma)$, but whose length is $O(\sigma^{-1})$. At the rear of the solitary wave the amplitude of the trailing shelf is σa_1^-, where $u_0 \to 0$ as $\theta \to -\infty$, but $u_1 \to u_1^-$. We readily find from (3.7) that

$$\frac{(V_0 - c)}{V_0} u_1^- = -\sqrt{c}\frac{\partial M_0}{\partial s}, \tag{3.16a}$$

where

$$M_0 = \int_{-\infty}^{\infty} \frac{V_0 u_0}{\sqrt{c}}\, \mathrm{d}\theta. \tag{3.16b}$$

Here M_0 is the leading-order term (with respect to σ) in the evaluation of the expression (2.2) for the mass of the solitary wave alone. Then, using (3.6) to evaluate M_0, we find that

$$\varepsilon^2 u_1^- = -\frac{6\sqrt{c}}{\mu a}\frac{\partial}{\partial s}\left\{\left(\frac{12\lambda|a|}{c|\mu|}\right)^{1/2}\right\}. \tag{3.17}$$

Note that u_1^- is proportional to ε^{-2}, so that the amplitude of the trailing shelf is proportional to $\beta = \sigma/\varepsilon^2$. Indeed it is readily verified that (3.17) agrees with the amplitude of the trailing shelf that is calculated directly from (1.1). However, here the variation of the amplitude $|a|$ depends explicitly on ε^2 (see (3.13)), and so consequently u_1^- will also depend explicitly on ε^2. To demonstrate the variation of u_1^-, we note that, using (3.14b)

$$\sqrt{\frac{c}{12}}M_0 = 2\,\mathrm{sign}\,\mu\left(\frac{\lambda A}{|\mu|}\right)^{1/3}. \tag{3.18}$$

Since, to leading order in ε^2, A is constant, it follows that the variation of M_0, and hence that of u_1^-, is largely determined by the combination $(\lambda/|\mu|)^{1/3}c^{-1/2}$. In particular, it is possible in principle for M_0 to increase or decrease independently of whether $|a|$ is increasing or decreasing (note that $|a|$ increases as $\lambda/|\mu|$ decreases to leading order in ε^2, but its variation is independent of c).

Finally we note that the first-order speed-correction term V_1 is not determined at this order, and it is necessary to proceed to second order to find it. The method for completing this calculation has been described in [11] for an equation equivalent to (1.1), and the same method could be applied here.

4. Conclusions

In this paper we have compared two alternative forms of the variable-coefficient KdV equation, namely the conventional form (1.1) and a more recent Hamiltonian form (1.4). Although a little more cumbersome in that it contains the governing small parameter ε^2 explicitly, the Hamiltonian form (1.4) should, in principle, be preferred, as it conserves the energy functional E (2.1) to a higher order of approximation than (1.1). However, the two equations are asymptotically equivalent, and indeed (1.5) demonstrates that the Hamiltonian form (1.4) is just (1.1) with the inclusion of those $O(\varepsilon^2)$ terms needed to ensure that the 'full' Hamiltonian (1.3) is conserved, rather than just its leading term. This, in fact, is the principal conclusion from our discussion of the conservation laws in Section 2.

In Section 3 we have considered the slowly-varying solitary-wave solution of the Hamiltonian form (1.4) in order to compare it with the well-known slowly-varying solitary wave solution of (1.1). As expected, the amplitude variation is determined by conservation of energy, and hence the result (3.14) obtained from (1.4) differs by $O(\varepsilon^2)$ terms from the equivalent result obtained from (1.1). These terms always act to reduce the variation of the amplitude.

However, in practice, it would seem that even for moderate values of ε (up to 0·6), the actual effect of the $O(\varepsilon^2)$ terms is quite small.

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